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The unsteady matched Stokes–Oseen solution for the flow past a sphere

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The authors present the matched asymptotic expansion type of solution for the unsteady viscous incompressible flow past a sphere. Most of the analysis is developed under the assumption that a constant rectilinear velocity is suddenly imparted to a sphere in an otherwise quiescent infinite body of fluid. The Reynolds number based on that velocity is taken to be small, and the analysis is then extended to other transient flows that satisfy this requirement. Evidently in the unsteady cases under discussion one can recognize inner and outer regimes. The leading terms in the expansions representing the flow in these are governed by the unsteady Stokes and the hitherto unreported unsteady Oseen equations. The streamline patterns calculated show the 'birth' of a ring vortex close to the equator and its gradual migration downstream and outwards. This result is also verified qualitatively by a crude experiment.

1. Introduction

The authors solve for the unsteady viscous incompressible flow past a solid sphere when a finite rectilinear velocity U is suddenly imparted to the sphere. The solution is obtained by the method of matched asymptotic expansions. Thus in the vicinity of the sphere the flow is governed by the unsteady Stokes equation; as pointed out below, this has already been studied. The outer flow field is found to be governed by a hitherto unreported relationship. It will be called the unsteady Oseen equation, because in the absence of time variations it reduces to the classical Oseen equation.

The solution obtained represents the entire process of transition from stagnancy to the steady state envisioned by Proudman & Pearson (1957). Throughout this process vorticity is generated at the solid surface and is transported away via convection and diffusion. Thus the streamline patterns in the outer field, presented in figures 2–4, show how the core of a ring vortex is 'born' close to the equator and then migrates downstream and outwards. When that core finally reaches infinity, one gets the well-known Oseen pattern shown in Schlichting (1960), Batchelor (1967) and other standard texts, which is in the form of an axial inflow through the wake that turns outwards as it reaches the sphere. Similarly, the streamline patterns in the inner field, depicted in figures 5(*a*)–(*d*), demonstrate the birth and outward migration of a vortex ring from the sphere to infinity. The latter case corresponds to the symmetrical steady Stokes flow past a sphere.

Since the motion starts from rest and since the unsteady Stokes and Oseen equations are linear, Laplace transform methods are used to account for the transience in both. However, the time co-ordinate is differently scaled in the two domains. Therefore a new procedure is devised to match the transforms associated with the inner and outer fields.

The Laplace transform of the Stokes solution yields the transform of the drag force, from which its time dependence can be obtained by inversion. The drag coefficient is singular at the beginning of the motion and eventually reaches the value $(6\pi)/R_e$ where R_e is the Reynolds number based on the velocity U and radius a of the sphere. We then generalize the above-mentioned transform relationship to obtain the time dependence of the drag for a sphere moving at an arbitrary speed or the speed for an arbitrary time-dependent applied drag. This treatment is believed to be more concise than that developed in Yih's (1969) text, which is based on Fourier transform methods. This advantage is demonstrated by the immediate solution for the case of a sphere released from rest. However, we stress that these results are only by-products of the Stokes analysis. Our major aim is to present a picture of the entire flow field.

2. Analysis

In terms of dimensional time and space co-ordinates fixed to the sphere, the flow problem is governed by the following differential equation:

$$\frac{\partial}{\partial t'}(d^2\psi')+\frac{1}{r'^2\sin\theta}\frac{\partial(\psi',d^2\psi')}{\partial(r',\theta)}+\frac{2d^2\psi'}{r'^3\sin^3\theta}\frac{\partial(\psi',r'\sin\theta)}{\partial(r',\theta)}=\nu d^4\psi'. \tag{1}$$

Here ν is the kinematic viscosity and the operator d^2 is defined by

$$d^2=(\partial/\partial r')^2+r'^{-2}\sin\theta(\partial/\partial\theta)(\sin\theta)^{-1}(\partial/\partial\theta),$$

where (r',θ,ϕ) are spherical co-ordinates with $r'=0$ at the centre of the sphere. The stream function ψ' is related to the velocity components in the (r',θ) directions as follows:

$$u'_r=(r'^2\sin\theta)^{-1}\partial\psi'/\partial\theta,\quad u'_\theta=-(r'\sin\theta)^{-1}\partial\psi'/\partial r'.$$

The boundary and initial conditions satisfied by ψ' are

$$\psi'=\partial\psi'/\partial r'=0\quad\text{for}\quad r'=a,\quad 0\leqslant\theta\leqslant\pi, \tag{2),(3}$$

$$\psi'\sim-(\tfrac{1}{2})Ur'^2\sin^2\theta H(t')\quad\text{as}\quad r'\to\infty, \tag{4}$$

$$\psi'=0\quad\text{for}\quad a\leqslant r'<\infty,\quad t'=0. \tag{5}$$

Here U is the linear velocity imparted to the sphere and $H(t')$ is the Heaviside step function. Thus H is unity for $t'>0$ and zero otherwise. We assume axisymmetric flow and vanishing velocity in the ϕ direction. Condition (4) implies that the entire expanse of liquid undergoes acceleration. It is represented by body-force terms in the Navier–Stokes equations. However these body forces are uniform thus, like gravity, they do not affect the flow kinematics.

It was shown by Proudman & Pearson (1957) that for steady-state motion the radial co-ordinate is differently scaled close and far from the sphere. The proposed analysis is based on a generalization of this assumption. Thus R and T are the radial and time co-ordinates in the far field, where they are defined in terms of length and time scales which are independent of a. The corresponding lower case co-ordinates r and t represent space and time variations in the near field. The stream function ψ is approximated by different expansions in the inner and outer domains. Arguments are presented below which support the choices made, but it is the successful matching which is taken as an indication that the co-ordinate scalings and expansions assumed are correct.

Let the dependent and independent variables be normalized thus:

$$\psi = \psi'/Ua^2, \quad r = r'/a, \quad R = r'U/\nu = rR_e, \quad t = t'\nu/a^2, \quad T = t'U^2/\nu = tR_e^2,$$

where R_e is the Reynolds number, defined by $R_e = Ua/\nu$. It follows from the governing equation that close to the sphere the stream function is governed by

$$(\partial/\partial t - \Delta^2)\,\Delta^2\psi^{(i)} = R_e(\text{conv}), \tag{6}$$

where the bracketed superscript indicates 'inner'. Since this equation is used to generate just the first term of the inner expansion, the convective terms on the right-hand side need not be given explicitly. The non-dimensional operators appearing in (6) and below are defined by

$$\Delta^2 \equiv a^2d^2, \quad D^2 \equiv (\nu/U)^2d^2 = R_e^{-2}\,\Delta^2.$$

For the outer field the following holds:

$$-(\partial/\partial T - D^2)\,D^2\psi^{(o)} = R_e^2\left\{\frac{1}{R^2\sin\theta}\,\frac{\partial(\psi^{(o)}, D^2\psi^{(o)})}{\partial(R,\theta)} + \frac{2D^2\psi^{(o)}}{R^3\sin^2\theta}\,\frac{\partial(\psi^{(o)}, R\sin\theta)}{\partial(R,\theta)}\right\}. \tag{7}$$

Numerous studies suggest that the inner expansion has the form

$$\psi^{(i)} = \psi_0^{(i)}(r,\theta,t) + O(R_e^1) \tag{8}$$

and that the first term is $\sin^2\theta$ times a function of r and t. The Laplace transform method is used to evaluate $\psi_0^{(i)}$ as well as the leading component of the outer expansion which is developed below. We let $\Psi_j^{(i)}(r,\theta,s)$ and $\Psi_k^{(o)}(R,\theta,S)$ be the transforms of $\psi_j^{(i)}(r,\theta,t)$ and $\psi_k^{(o)}(R,\theta,T)$, respectively. In view of the initial condition (5) and the governing equation (6), $\Psi_0^{(i)}$ satisfies

$$(s-\Delta^2)\,\Delta^2\Psi_0^{(i)} = 0. \tag{9}$$

For the assumed dependence on θ the most general solution of this equation is

$$\Psi_0^{(i)} = \sin^2\theta\{A(s)\,r^2 + B(s)\,r^{-1} + C(s)\,r^{\frac{1}{2}}K_{\frac{3}{2}}(s^{\frac{1}{2}}r) + D(s)\,r^{\frac{1}{2}}I_{\frac{3}{2}}(s^{\frac{1}{2}}r)\}, \tag{10}$$

where A, B, C and D are functions of s to be determined and I and K denote the modified Bessel functions of the first and second kind respectively.

In view of (4), $D(s)$ vanishes while $A(s)$ is $-\frac{1}{2}s^{-1}$. $B(s)$ and $C(s)$ can be found by invoking (2) and (3). It is consequently found that the transform is given by

$$\Psi_0^{(i)} = \sin^2\theta\left\{-\tfrac{1}{2}r^2\frac{1}{s} + \frac{1}{2rs^2}(3+3s^{\frac{1}{2}}+s) - \frac{3}{2rs^2}(1+sr^{\frac{1}{2}})\exp\left[-s^{\frac{1}{2}}(r-1)\right]\right\}. \tag{10a}$$

Its inverse is

$$\psi_0^{(i)} = \sin^2\theta\left\{-\tfrac{1}{2}r^2H(t) + \frac{1}{2r}\left[H(t) + 3\left(\frac{4t}{\pi}\right)^{\frac{1}{2}} + 3t\right] - \tfrac{3}{2}t^{\frac{1}{2}}\left[\left(\frac{4}{\pi}\right)^{\frac{1}{2}}\exp(-\eta^2) - 2\eta\operatorname{erfc}\eta\right]\right.$$
$$\left.-\frac{3}{2}\frac{t}{r}\left[(1+2\eta^2)\operatorname{erfc}\eta - \left(\frac{4}{\pi}\right)^{\frac{1}{2}}\eta\exp(-\eta^2)\right]\right\}, \tag{11}$$

where $$\eta \equiv (r-1)(4t)^{-\frac{1}{2}}.$$

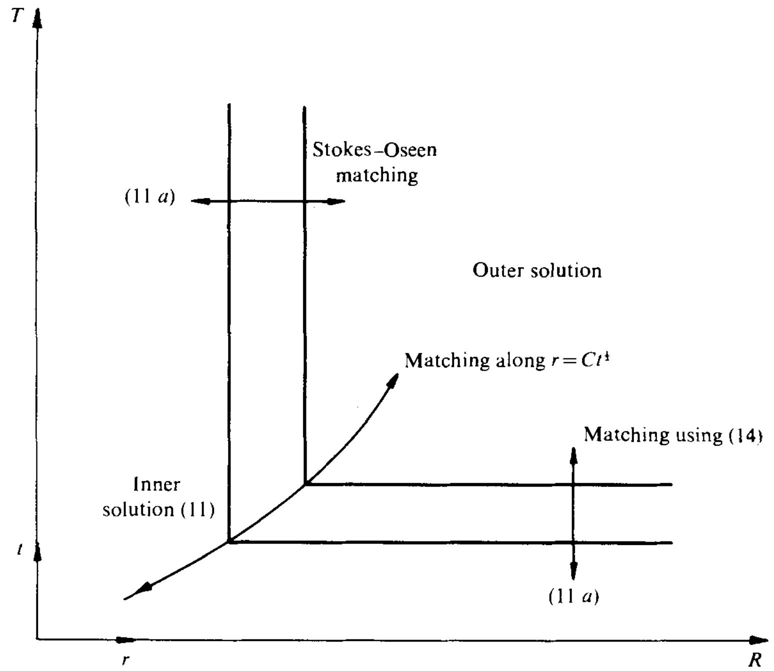


FIGURE 1. Schematic sketch demonstrating the matching procedure between the inner and outer solutions.

Expansion (8) can be expected to hold so long as the amount of vorticity in the field is not too large. Thus in the steady Stokes solution which is eventually attained, i.e.

$$\psi_0^{(i)} \sim \sin^2\theta\{-\tfrac{1}{2}r^2 + \tfrac{3}{4}r - \tfrac{1}{4}r^{-1}\} \quad \text{as} \quad t \to \infty, \tag{11a}$$

the rotational component increases like r. Indeed it was shown by Whitehead that if (6) is used to generate $\psi_1^{(i)}(r, \theta, \infty)$ then the vorticity convection gives rise to an unbounded particular integral. However, the most important part of the process under discussion is the period before an appreciable amount of vorticity is generated. Thus for any finite t, $\psi_0^{(i)}$ can be approximated by

$$\psi_0^{(i)} \sim \sin^2\theta\{-H(t)\tfrac{1}{2}r^2 + \tfrac{1}{2}r^{-1}[H(t) + 3(4t/\pi)^{\frac{1}{2}} + 3t]\} \quad \text{as} \quad r \to \infty. \tag{11b}$$

This represents irrotational flow; the rotational components of $\psi_0^{(i)}$ and hence also the higher-order terms of (8) decay like $\exp(-\gamma r^2)$, where γ is a constant. This implies that the vorticity is restricted to the vicinity of the sphere and that for finite t expansion (8) is valid throughout the flow field.

These features are demonstrated in figure 1, which is also useful in explaining the ensuing analysis. In the L-shaped region adjacent to the r and t axes, the inner solution is valid. (This is symbolic. We do not suggest that the region of validity has this precise geometry.) Away from this region the inner solution is singular and there the outer solution, which is constructed below, prevails. It is assumed that the domains of validity of the two overlap as shown.

Closer examination of (11) reveals that the rotational part of $\psi_0^{(i)}$ becomes sizable if both r and t are increased indefinitely along the path $r = Ct^{\frac{1}{2}}$, where C is a positive finite constant. At the end of these paths (8) ceases to hold and it is in this region that an outer expansion $\psi^{(o)}$ is constructed. We assume that the outer radial co-ordinate is that adopted by Proudman & Pearson; their solution is recovered here as a special case. Once the radial 'stretching ratio' has been set, the time co-ordinate follows from the nature of the paths in the r, t domain leading to the region of non-uniformity.

We now solve for the stream function in the outer domain. The appropriate expansion is

$$\psi^{(o)} \sim -\tfrac{1}{2}R_e^{-2}R^2\sin^2\theta + R_e^{-1}\psi_{-1}^{(o)}(R,\theta,T), \quad T > 0. \tag{12}$$

This can be deduced by recasting the solution for $\psi^{(i)}$ in terms of R and T and then rearranging it as a series in descending orders of R_e. Thus, as might be expected, the term representing a uniform irrotational stream dominates. However, it is the deviation from uniformity and irrotationality which is of interest. It can indeed be shown that the first term in (12) satisfies (7). The term representing the deviation $O(R_e^{-1})$ is governed by the hitherto unreported relationship

$$(D^2 + \partial/\partial X - \partial/\partial T)\,D^2\psi_{-1}^{(o)} = 0, \tag{13}$$

in which X is the Cartesian co-ordinate defined by $(X, \Omega) = R(\cos\theta, \sin\theta)$. This will be called the unsteady Oseen equation. Unlike its steady counterpart, it represents not only the diffusion and convection of vorticity, but also temporal variations in that quantity. Indeed (5) implies that there is no vorticity initially. It is generated as the process progresses. If it is assumed that a steady state is eventually attained, then $\psi^{(o)}(R,\theta,T)$ approaches the Oseen solution as T is increased.

As a first step in the construction of the solution for the stream function, the outer-field vorticity is evaluated. As explained, for any finite t the vorticity is negligible in the outer field. Hence the following initial contition will be assumed to hold:

$$D^2\psi_{-1}^{(o)}(R,\theta,0) = 0. \tag{14}$$

By Laplace transforming (13) one gets

$$(D^2 + \partial/\partial X - S)\,D^2\Psi_{-1}^{(o)} = 0. \tag{15}$$

Matching considerations which are invoked below suggest that $D^2\Psi_{-1}^{(o)}$ is proportional to $\sin^2\theta$ times a function of R and S. The most general solution of (15) that has this form and vanishes at infinity is

$$D^2\Psi_{-1}^{(o)} = P(S)\exp\left[-\tfrac{1}{2}R(\zeta-\mu)\right](1 + 2/R\zeta)(1-\mu^2), \tag{16}$$

where $\mu \equiv \cos\theta$, $\zeta \equiv (4S+1)^{\frac{1}{2}}$ and $P(S)$ is a constant. The last quantity is evaluated by noting that the relationship

$$\int_0^\infty (\ )\exp(-st)\,dt = \int_0^\infty R_e^{-2}(\ )\exp(-sTR_e^{-2})\,dT = \int_0^\infty R_e^{-2}(\ )\exp(-ST)\,dT$$

holds between the two types of Laplace transform and that it is the same dependent variable ψ which is sought in both the inner and the outer field. This implies that the following relationships hold:

$$SR_e^2 = s, \quad \Psi^{(i)}(r,\theta,s) = \Psi^{(o)}(R/R_e,\theta,S)\,R_e^{-2}. \tag{17),(18}$$

The last two equations together with the relationship $R = rR_e$ enable one to match the right-hand side of (16) with the Laplace transform of the vorticity associated with the inner field, as given by (10*a*). The resulting solution for the vorticity is thus found to be

$$D^2\Psi_{-1}^{(o)} = -\frac{3}{4}\frac{\zeta}{S}\exp\left[-\frac{R}{2}(\zeta-\mu)\right]\left(1+\frac{2}{R\zeta}\right)(1-\mu^2). \tag{16a}$$

Note that the matching requirements as enforced do not yield a unique solution for $P(S)$. For example, these could be satisfied by setting $P(S) = -3/(4S)^{\frac{1}{2}}$. This non-uniqueness reflects the fact that the matching expressed in terms of (R, r) and (S, s) is really applied along the paths $r = Ct^{\frac{1}{2}}$ shown in figure 1. However, the outer-field vorticity is a solution of an initial-value problem. It embodies the initial condition (14), which was invoked as well as one end condition, namely the requirement that the vorticity should be finite for $R\to\infty$ at any value of T. The matching along the path $r = Ct^{\frac{1}{2}}$ does not reflect the other end condition. In particular, it does not ensure that the outer-field vorticity matches that associated with the inner field for $t\to\infty$, i.e. that the outer Oseen flow matches the steady inner Stokes solution. By employing the well-known property of Laplace transforms that

$$\lim_{S\to 0} S\Psi_{-1}^{(o)}(R,\theta,S) = \psi_{-1}^{(o)}(R,\theta,\infty),$$

one can show that with our choice of $P(S)$ as given by (16*a*) this requirement is satisfied.

Rather than go through the lengthy process of integrating (16*a*) we shall show that the solution

$$\begin{aligned}\Psi_{-1}^{(o)} = {} & \frac{3}{2S}\left[1+\mu-\exp\left[-\frac{R}{2}(\zeta-\mu)\right]\left(\frac{2S+1}{\zeta}+\mu\right)\right]+\frac{12S}{\zeta(\zeta+1)^2}\exp\left[-\frac{R}{2}(\zeta+1)\right]\\ & -\frac{3S}{2\zeta}\sum_{n=0}^{\infty}\left(\int_{-1}^{\mu}P_n(\hat{\mu})\,d\hat{\mu}\right)\left\{\int_0^R \exp\left(-\frac{\xi}{2}\zeta\right)\left(\frac{\pi}{\xi}\right)^{\frac{1}{2}} I_{n+\frac{1}{2}}\left(\frac{\xi}{2}\right)\frac{\xi^{n+2}}{R^n}\,d\xi\right.\\ & \left.+\int_R^{\infty}\exp\left(-\frac{\xi}{2}\zeta\right)\left(\frac{\pi}{\xi}\right)^{\frac{1}{2}} I_{n+\frac{1}{2}}\left(\frac{\xi}{2}\right)\frac{R^{n+1}}{\xi^{n-1}}\,d\xi\right\}\end{aligned} \tag{19}$$

meets the conditions of the problem. Here the $I_{n+\frac{1}{2}}$ are modified Bessel functions of the first kind while the P_n are Legendre polynomials. The integrals of the latter functions satisfy

$$\sin\theta(\partial/\partial\theta)(\sin\theta)^{-1}\partial/\partial\theta\left(\int_{-1}^{\mu}P_n(\hat{\mu})\,d\hat{\mu}\right) = -n(n+1)\left(\int_{-1}^{\mu}P_n(\hat{\mu})\,d\hat{\mu}\right), \tag{20}$$

where the differential operator in θ appears also in the definition of D^2. Therefore substituting from (19) into (16*a*) shows that the latter is satisfied. In this verification, use is made of the relationship

$$2[\exp(\tfrac{1}{2}R\mu)-\exp(-\tfrac{1}{2}R)] = (\pi R)^{\frac{1}{2}}\sum_{n=0}^{\infty}(2n+1)I_{n+\frac{1}{2}}(\tfrac{1}{2}R)\left(\int_{-1}^{\mu}P_n(\hat{\mu})\,d\hat{\mu}\right), \tag{21}$$

which is obtained by straightforward integration of the well-known equation

$$\exp(\tfrac{1}{2}R\mu) = (\pi/R)^{\frac{1}{2}}\sum_{n=0}^{\infty}(2n+1)I_{n+\frac{1}{2}}(\tfrac{1}{2}R)P_n(\mu). \tag{22}$$

It is then noted that for $\mu = -1$ every term in the summation vanishes while the other two expressions on the right-hand side of (19) cancel one another. For $\mu = 1$

all terms in the summation vanish except the first. However, for $n = 0$, the integrals with respect to ξ inside the curly brackets can be easily evaluated by expressing $I_{\frac{1}{2}}(Z)$ as $(2\pi/Z)^{\frac{1}{2}} \sinh Z$. It can thus be shown that the transform satisfies

$$\Psi_{-1}^{(o)}(R, 0, S) = \Psi_{-1}^{(o)}(R, \pi, S) = 0, \qquad (23), (24)$$

which implies that the axis of symmetry constitutes a streamline.

By inverting the right-hand side of (19) one gets

$$\begin{aligned}
\psi_{-1}^{(o)} &= \tfrac{3}{2}(1+\mu)\,H(t) - \tfrac{3}{2}(\pi T)^{-\frac{1}{2}} \exp\left(\tfrac{1}{2}\mu R - \tfrac{1}{4}T - R^2/4T\right) \\
&\quad - \tfrac{3}{4}\exp\left(\tfrac{1}{2}\mu R\right) \int_0^T (\pi S)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{4}S - R^2/4S\right) dS \\
&\quad - \tfrac{3}{4}\mu R \exp\left(\tfrac{1}{2}\mu R\right) \int_0^T (\pi S)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{4}S - R^2/4S\right) dS/S \\
&\quad + \tfrac{3}{4}\exp\left(-\tfrac{1}{2}R\right) \frac{d}{dT}\{\exp\left(-\tfrac{1}{4}T\right)[2(T/\pi)^{\frac{1}{2}} \exp\left(-R^2/4T\right) \\
&\quad - (T+R)\exp\left(\tfrac{1}{2}R + \tfrac{1}{4}T\right) \operatorname{erfc}\left(\tfrac{1}{2}RT^{-\frac{1}{2}} + \tfrac{1}{2}T^{\frac{1}{2}}\right)]\} \\
&\quad - \frac{3}{4}\sum_{n=0}^{\infty}\left(\int_{-1}^{\mu} P_n(\hat{\mu})\, d\hat{\mu}\right)\frac{d}{dT}\left\{(\pi T)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{4}T\right)\left[\int_0^R (\pi/\xi)^{\frac{1}{2}} I_{n+\frac{1}{2}}\left(\tfrac{1}{2}\xi\right) \exp\left(-\xi^2/4T\right)\right.\right. \\
&\quad \left.\left. \times (\xi/R)^n \xi^2 d\xi + R^2 \int_R^{\infty} (\pi/\xi)^{\frac{1}{2}} I_{n+\frac{1}{2}}\left(\tfrac{1}{2}\xi\right) \exp\left(-\xi^2/4T\right) (R/\xi)^{n-1} d\xi\right]\right\} \\
&= \tfrac{3}{2}\mu H(T) + \tfrac{3}{4}(1-\mu)\operatorname{erf}\left[(R^2/4T)^{\frac{1}{2}} + \left(\tfrac{1}{4}T\right)^{\frac{1}{2}}\right]\{1 - \exp\left[\tfrac{1}{2}R(1+\mu)\right]\} \\
&\quad - \tfrac{3}{4}(1+\mu)\operatorname{erf}\left[(R^2/4T)^{\frac{1}{2}} - \left(\tfrac{1}{4}T\right)^{\frac{1}{2}}\right]\{1 - \exp\left[-\tfrac{1}{2}R(1-\mu)\right]\} \\
&\quad + \tfrac{3}{2}\exp\left(\tfrac{1}{2}\mu R\right)\left[\sinh \tfrac{1}{2}R - \mu \cosh \tfrac{1}{2}R\right] \\
&\quad - \tfrac{3}{2}(\pi T)^{-\frac{1}{2}} \exp\left(-R^2/4T - \tfrac{1}{4}T\right)\left[\exp\left(\tfrac{1}{2}\mu R\right) - \cosh \tfrac{1}{2}R - \mu \sinh \tfrac{1}{2}R\right] \\
&\quad - \tfrac{3}{4}(RT)^{-\frac{1}{2}} R^3 \exp\left(-\tfrac{1}{4}T\right) \sum_{n=1}^{\infty} (2n+1)^{-1}[P_{n+1}(\mu) - P_{n-1}(\mu)]\, F_n(R, T),
\end{aligned} \qquad (25)$$

where

$$\begin{aligned}
F_n(R, T) &= \int_0^1 \{(R^2\xi^2/4T^2 - \tfrac{1}{2}T - \tfrac{1}{4})\, I_{n+\frac{1}{2}}\left(\tfrac{1}{2}R\xi\right) \exp\left(-R^2\xi^2/4T\right) \xi^{n+\frac{3}{2}} \\
&+ (R^2/4\xi^2T^2 - \tfrac{1}{2}T - \tfrac{1}{4})\, I_{n+\frac{1}{2}}(R/2\xi) \exp\left(-R^2/4T\xi^2\right) \xi^{n-\frac{5}{2}}\}\, d\xi.
\end{aligned}$$

The reader can easily check that the solution satisfies the following relationships:

$$\lim_{T\to\infty} \psi_{-1}^{(o)} = \tfrac{3}{2}(1+\mu)\{1 - \exp\left[-\tfrac{1}{2}R(1-\mu)\right]\}, \quad \lim_{T\to 0} \psi_{-1}^{(o)} = 0.$$

Consequently it indeed represents the transition from stagnancy to the steady flow field described by Proudman & Pearson's solution.

3. Discussion

By integrating the stresses over the sphere one gets the following well-known expression for the drag:

$$f(t) = \frac{\rho U^2 a^2}{R_e}\int_0^{\pi}\left\{\left[p - 2\frac{\partial u_r}{\partial r}\right]\cos\theta + \left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]\sin\theta\right\}_{r=1} 2\pi \sin\theta\, d\theta. \qquad (26)$$

In this equation ρ is the fluid density while u_r and u_θ are the dimensionless velocity components in the indicated directions normalized with respect to U. The pressure p is normalized with respect to $\rho U^2/a$.

Clearly it is the inner-field values which should be substituted into the right-hand side of (26) in order to evaluate $f(t)$. Within the framework of the approximation adopted, u_r and u_θ are found as space derivatives of the only available component of $\psi^{(i)}$, namely $\psi_0^{(i)}$. The pressure term p is taken to be the sum $\hat{p}+\tilde{p}$, where $\hat{p}$ is the solution of

$$\frac{\partial \hat{p}}{\partial r} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial t} - \Delta^2\right)\psi_0^{(i)}, \tag{27}$$

$$\frac{1}{r}\frac{\partial \hat{p}}{\partial \theta} = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} - \Delta^2\right)\psi_0^{(i)} \tag{28}$$

and $\tilde{p}$ is the pressure field balancing the inertia of the entire expanse of liquid accelerating with respect to the sphere. It follows from these relationships together with (6) that only the irrotational component of $\psi_0^{(i)}$ contributes to the pressure term in the integral of (26), the rotational component of $\psi_0^{(i)}$ contributing only to the other terms in that integral. The integration is more easily carried out in terms of the transform $\Psi_0^{(i)}$ rather than $\psi_0^{(i)}$. This yields for $F(s)$, the transform of $f(t)$,

$$F(s) = (2\pi)\frac{\rho U^2 a^2}{R_e s}(3+3s^{\frac{1}{2}}+s) \tag{29}$$

for the case of sphere at rest which is suddenly given a velocity U. When the velocity is $v(t)\,U$ the relationship between its transform $V(s)$ and that of the force is

$$F(s) = (2\pi)\frac{\rho U^2 a^2}{R_e}(3+3s^{\frac{1}{2}}+s)\,V(s). \tag{29a}$$

This embodies a large class of drag–velocity relationships. Two of these will be recovered here.

When the time-dependent velocity is represented by a step function, the force is given by

$$f(t) = \frac{6\pi U^2 a^2}{R_e}\left\{H(t)+\frac{\delta(t)}{3}+\left(\frac{1}{\pi t}\right)^{\frac{1}{2}}\right\}, \tag{30}$$

where $\delta(t)$ denotes the Dirac delta function.

Conversely, when the force is represented by a step function the velocity is

$$v(t) = \frac{3U}{5^{\frac{1}{2}}}\left\{\frac{1}{\alpha}[1-\exp(\alpha^2 t)\operatorname{erfc}(\alpha t^{\frac{1}{2}})] - \frac{1}{\beta}[1-\exp(\beta^2 t)\operatorname{erfc}(\beta t^{\frac{1}{2}})]\right\}, \tag{31}$$

where U is the ultimate value of the velocity and is obtained from Stokes' relationship and the parameters α and β are given by

$$\alpha = \tfrac{3}{2}(3-5^{\frac{1}{2}}), \quad \beta = \tfrac{3}{2}(3+5^{\frac{1}{2}}).$$

We have demonstrated how easily (29a) yields the time-dependent drag or velocity when the other variable is a prescribed function of time. But this interrelationship holds only when $v(t)$ satisfies two conditions. First, its maximum value must be finite, so that the Reynolds number based on U is indeed small. Second, v must eventually

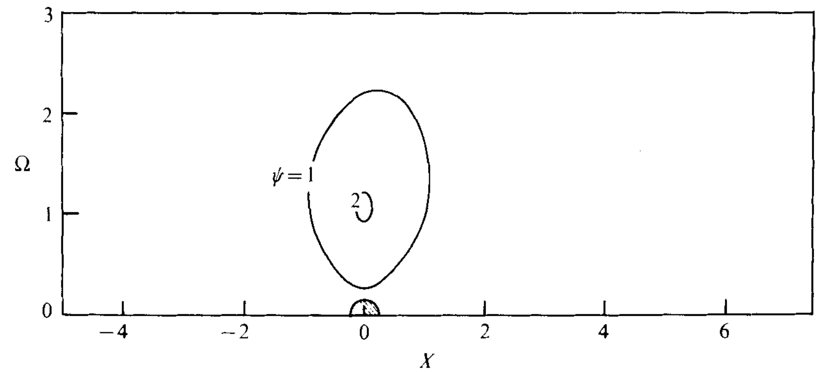


FIGURE 2. Birth of a ring vortex shortly after a sphere impulsively started from rest; $T = 0{\cdot}1$.

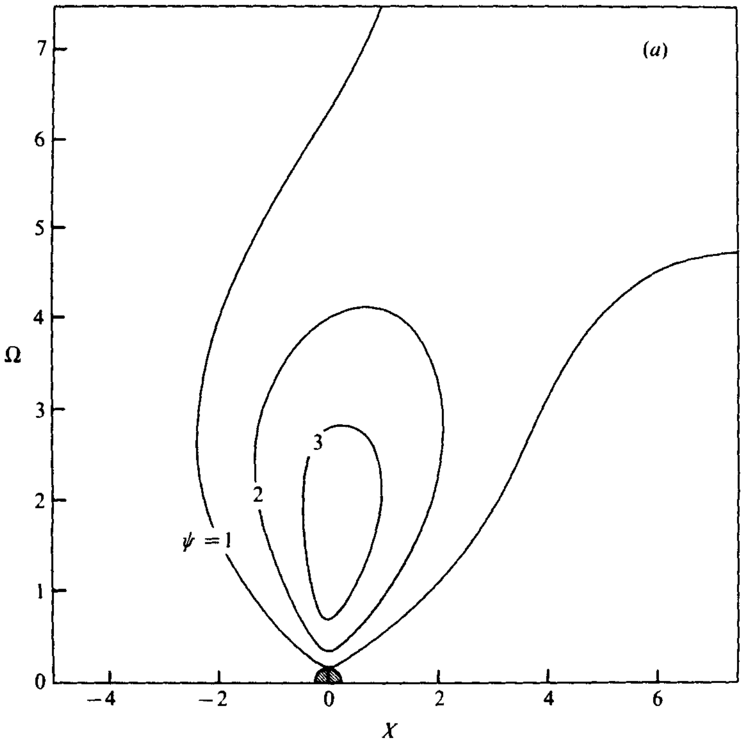
attain a terminal value of either zero or unity and the time scale τ characterizing the transition period must be $O(a^2/\nu)$ or smaller. To demonstrate the significance of the last requirement, let the dimensionless time be redefined thus: $t \equiv t'/\tau$. Owing to this rescaling, the first terms on the left-hand sides of (6) and (7), which represent the effect of transience, must be multiplied by $a^2/\tau\nu$. If τ is indeed small, these dominate the effect of convection, which is at most $O(R_e)$. In such cases, the ordering process underlying the proposed analysis is valid. It clearly ceases to be so if $a^2/\tau\nu$ is $O(R_e)$ because then the effects of convection and transience in (6) are of similar orders of magnitude.

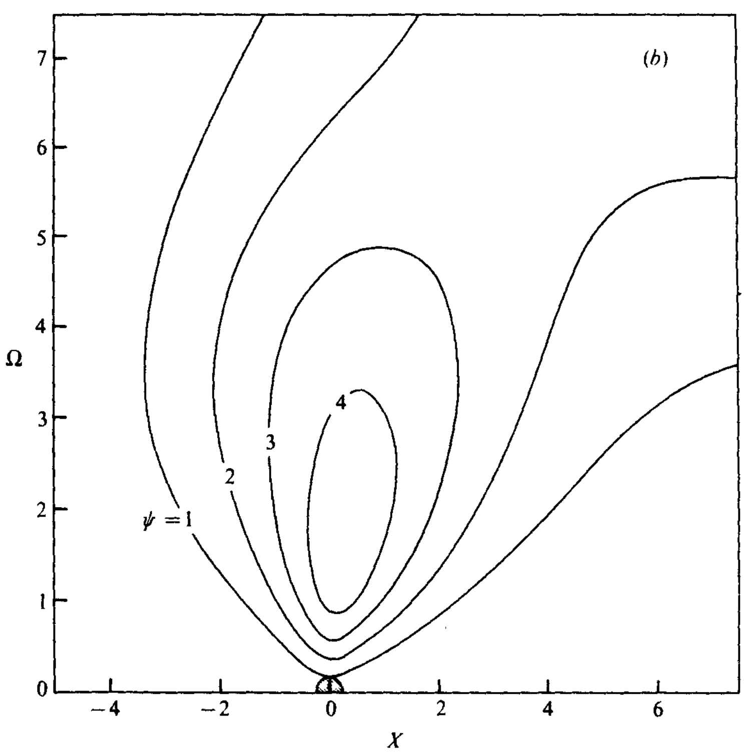
The second condition imposed on v affects the nature of the far field. Recasting the argument of v such that it reflects the second stipulation gives

$$v = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{for} \quad t \gg \tau\nu/a^2, \quad \text{whence} \quad v = \begin{Bmatrix} H(T) \\ 0 \end{Bmatrix},$$

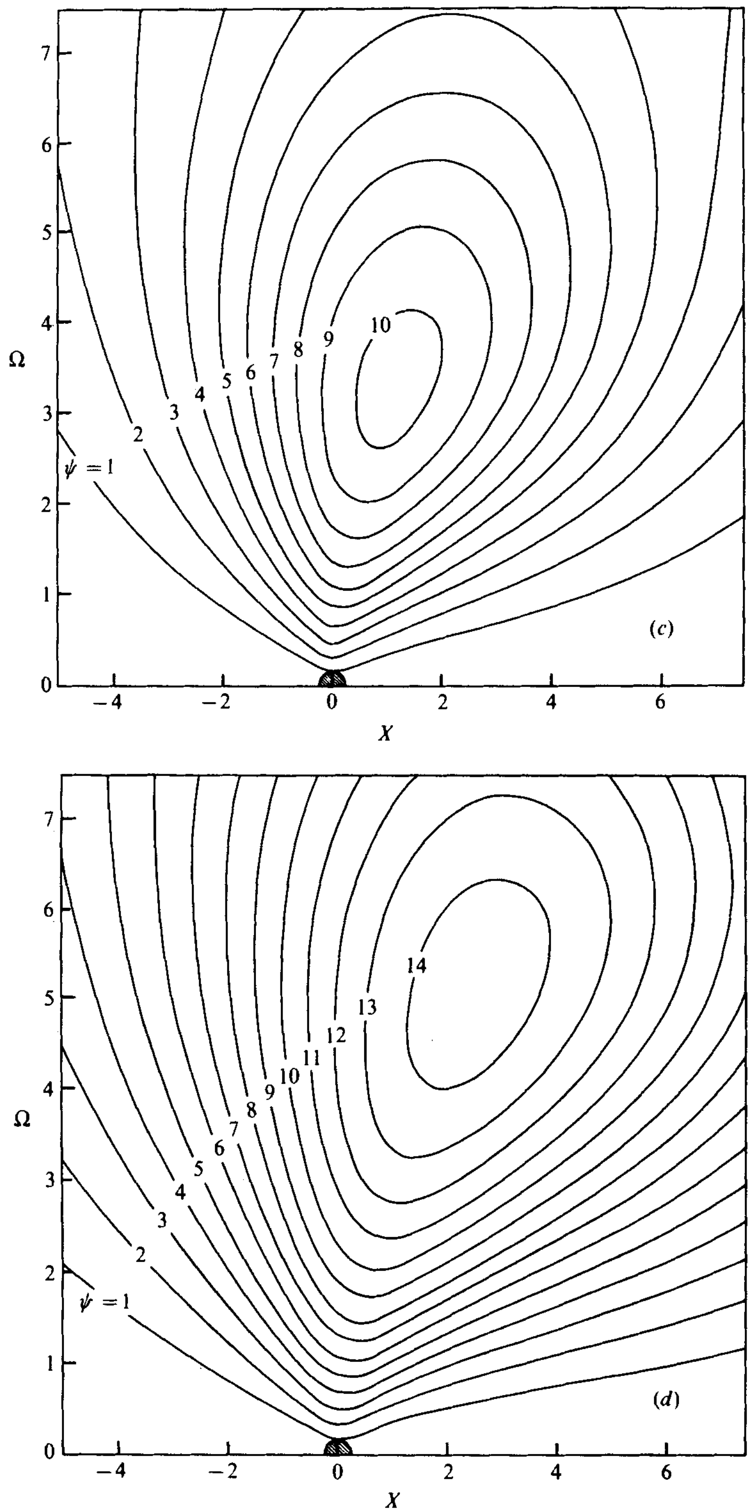
depending on the terminal value of v. Thus the outer solution constructed here for $v = H(t)$ holds for all cases in which the sphere attains a steady constant velocity. Hence the far-field solution expressed in terms of R and T is insensitive to the detailed manner in which that velocity is attained. The case of a suddenly applied force serves as an example. Indeed, when rewritten in terms of T, (31) expresses a step-function dependence, although in effect, or in terms of t, the terminal velocity is acquired gradually. On the other hand, if the sphere is moved and stopped within a time span $O(\tau)$, then $\psi^{(o)}$ vanishes identically. In such cases the amount of vorticity attained is finite and hence, as explained, the inner solution is non-singular throughout.

In figures 2–4 the streamline patterns of the disturbed outer flow, as represented by $\psi_{-1}^{(o)}$ and defined by (25), are plotted for $T = 0{\cdot}1, 0{\cdot}5, 1, 5, 10, 15, 20$ and for $T \to \infty$. Note that the flow pattern is independent of the Reynolds number. In these figures, the radius of the sphere is small and equal to the Reynolds number. Initially the disturbance is in the form of a vortex ring of low intensity located at the immediate vicinity of the sphere. As time progresses the vortex becomes more intense and its core moves downstream and away from the axis of symmetry. When ultimately a steady state is attained, as $T \to \infty$, the streamline pattern assumes the form envisaged by Oseen (figure 4). Here the core of the vortex ring is at infinity, the flow pattern at a large distance upstream resembling that for a simple source situated at the origin. To





FIGURES 3(*a*, *b*). For legend see p. 28.



Figures 3(c, d). For legend see next page.

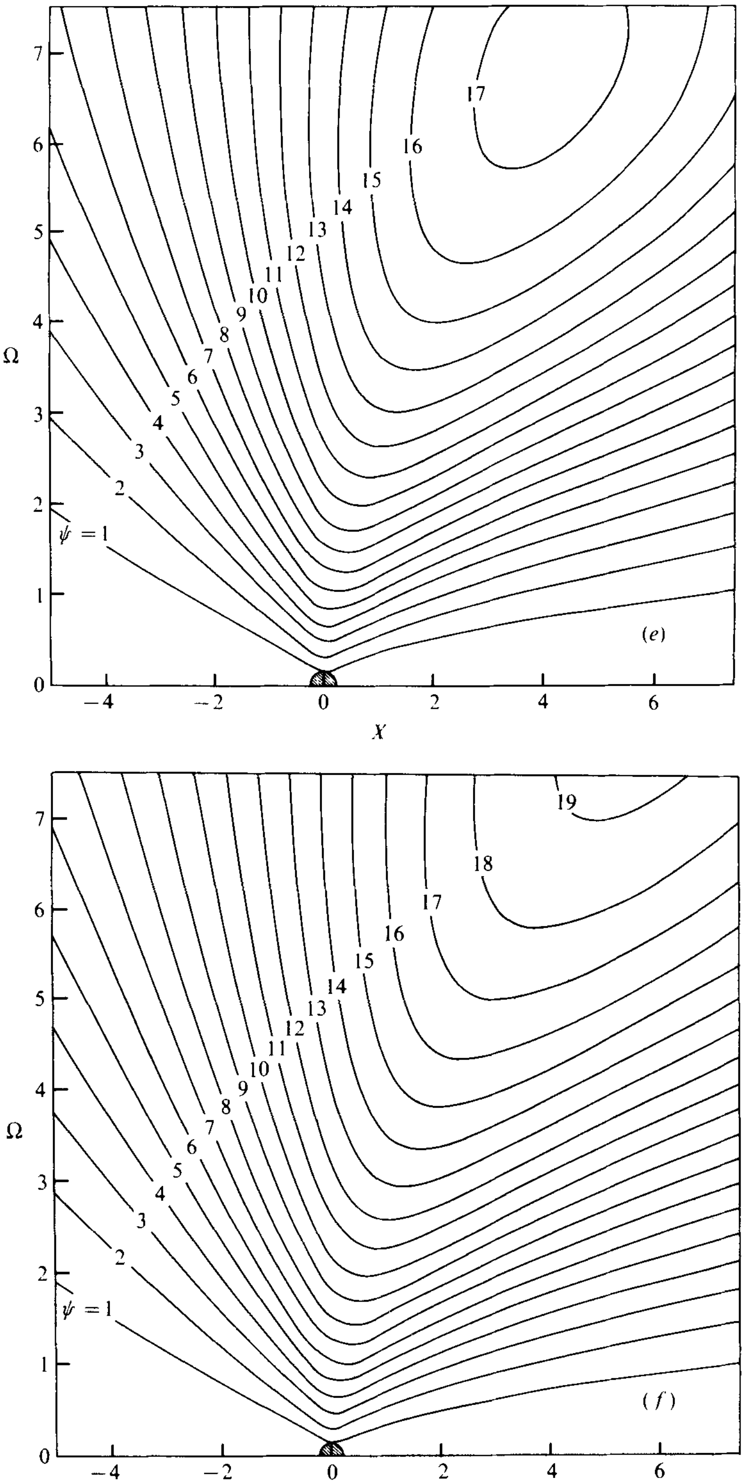


Figure 3. Streamline pattern in the outer field for transient flow past a sphere at (*a*) $T = 0{\cdot}5$, (*b*) $T = 1$, (*c*) $T = 5$, (*d*) $T = 10$, (*e*) $T = 15$ and (*f*) $T = 20$.

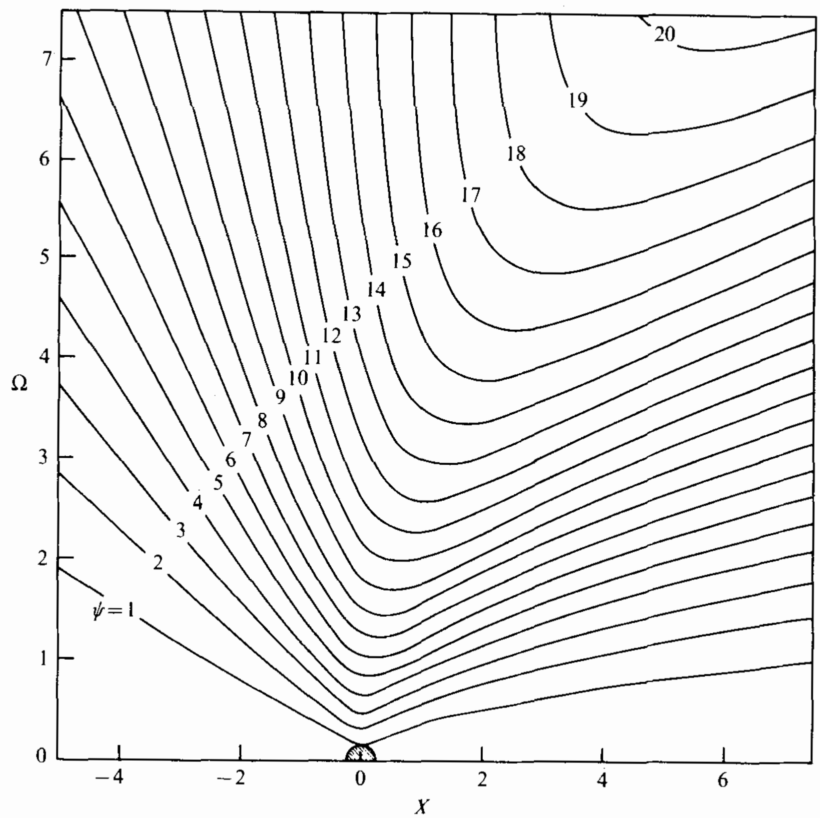
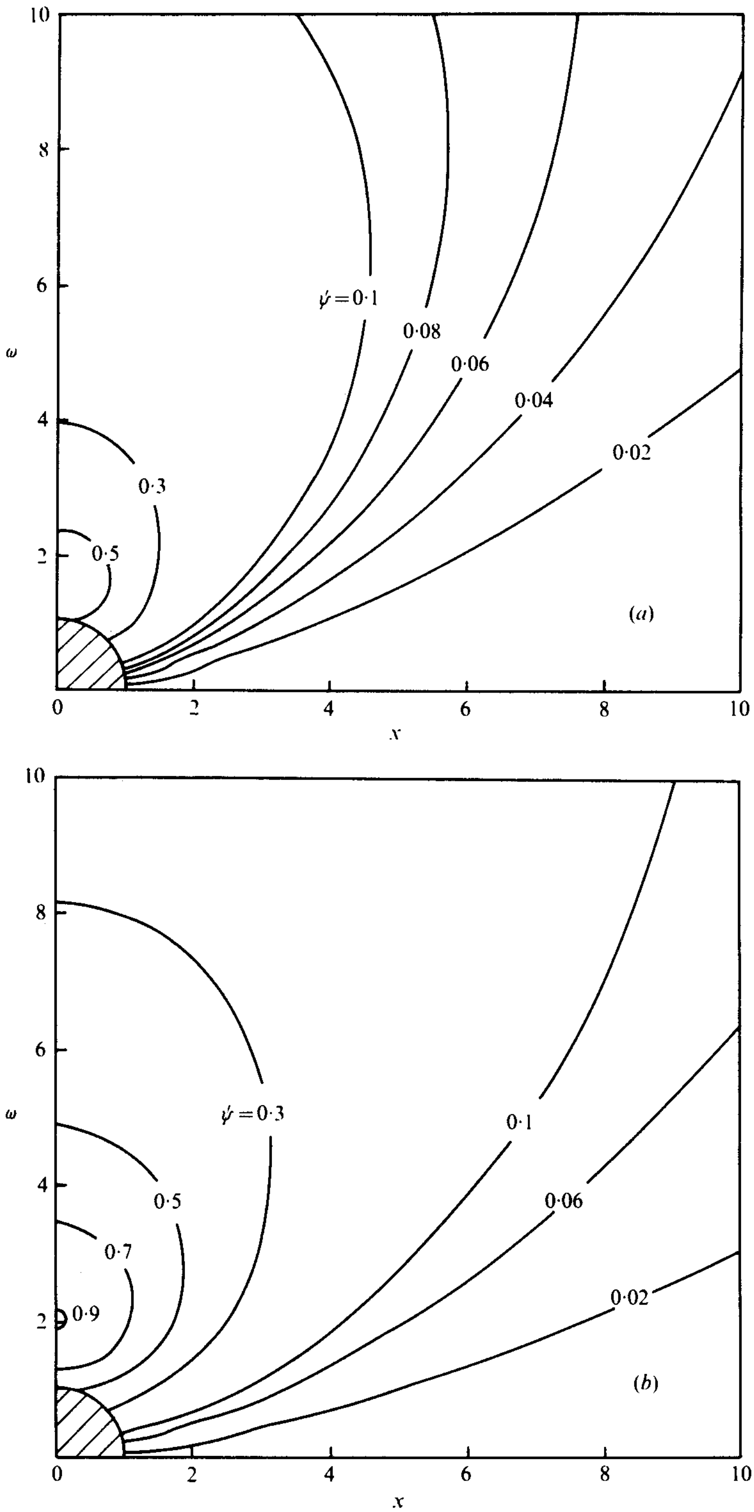


FIGURE 4. Streamline pattern of steady outer Oseen flow (as reproduced by Schlichting 1960).

compensate for such outward flow there is an inward flow towards the sphere downstream. The turning of the streamlines from outward flow downstream to inward flow upstream marks the boundary of a wake starting at the sphere and extending to infinity downstream.

In a similar manner the transience of the disturbed inner flow field $\psi_0^{(i)}$ given in (11) from stagnancy onwards is depicted in figures 5(*a*)–(*d*) for $t = 0{\cdot}1$, $0{\cdot}5$, 1 and 2. It should be noted that because of the nature of the inner solution, where convection is neglected, the streamlines are symmetric with respect to the plane $\theta = \frac{1}{2}\pi$ and do not depend on the direction of motion. For this reason it is sufficient to draw the flow field on only one quadrant. The streamline pattern is plotted in a co-ordinate system $(x, \omega) = r(\cos\theta, \sin\theta)$ where the sphere surface is defined by $r = 1$. For small t the streamline pattern demonstrates the 'birth' of a vortex ring at the sphere which migrates outwards along $\theta = \frac{1}{2}\pi$. For $t \to \infty$ the vortex reaches infinity and the flow field is given by the classical Stokes solution given in (11*a*) excluding the uniform-flow term.

We conclude this analytical presentation with a rather crude qualitative experiment which confirms some of the theoretical results. To simulate the model considered in the analysis, a metal sphere of radius 1 cm was suspended by an inextensible cord in a tank full of glycerine. At a certain instant, the cord was pulled sharply by hand such that the sphere instantaneously attained a velocity of approximately 0·8 cm/s.



FIGURES 5(a, b). For legend see facing page.

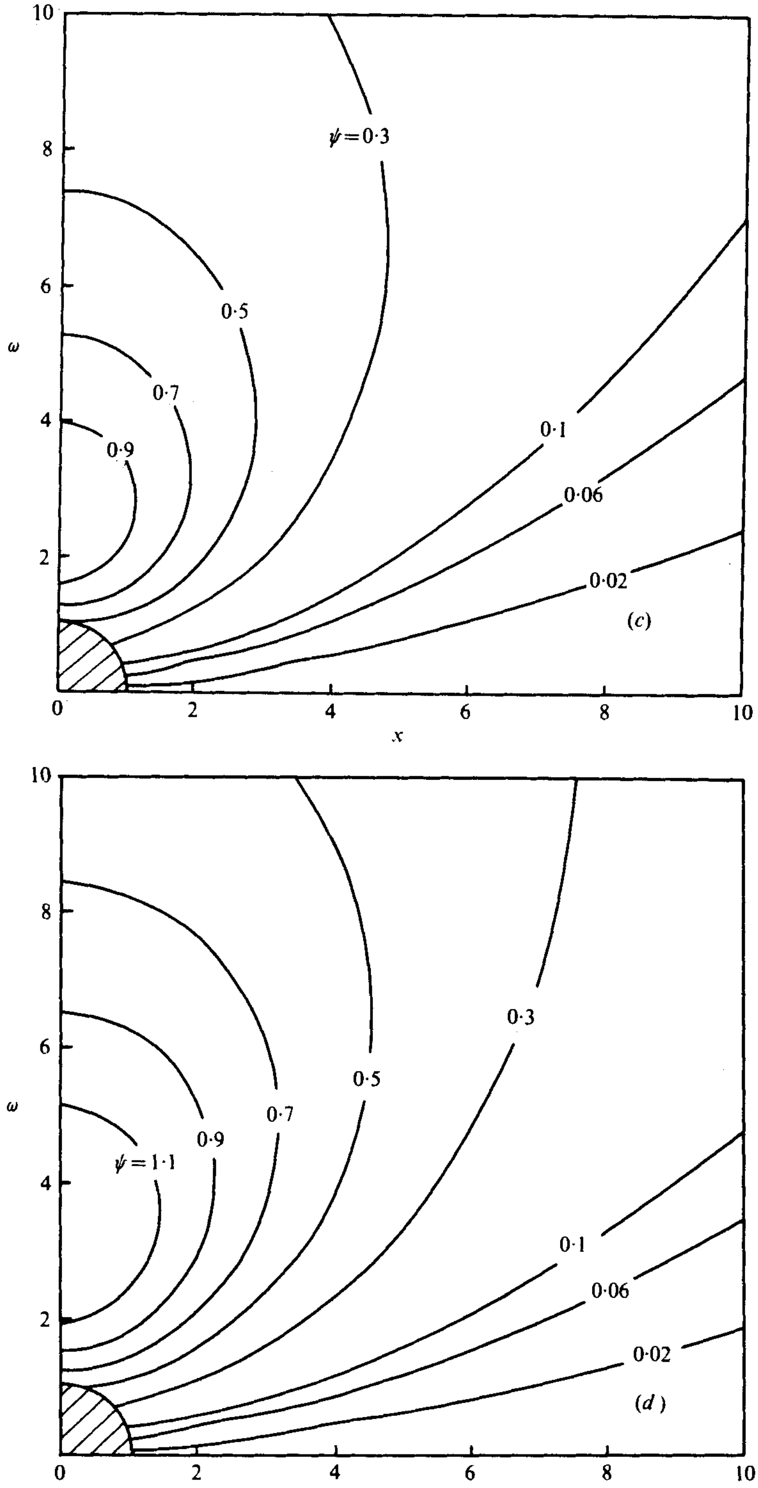


Figure 5. Streamline pattern in the inner field for transient flow past a sphere at (*a*) $t = 0{\cdot}1$, (*b*) $t = 0{\cdot}5$, (*c*) $t = 1$ and (*d*) $t = 2$.

The Reynolds number based on this velocity, the viscosity of the glycerine and the sphere diameter is about $R_e = 10^{-3}$. The glycerine contained air bubbles and the meridional plane was appropriately illuminated. By using these tracers in conjunction with a camera, it was possible to obtain the trajectories (pathlines) of the illuminated bubbles. A photograph taken immediately after motion commenced with an exposure time of 2 s is shown in figure 6 (plate 1). The resemblance between the analytical solution for the inner flow field and the crude experiment is obvious, both indicating outward migration of a vortex ring.

The experimental verification which transformed this work from a piece of speculative mathematics to realistic fluid mechanics was carried out by our colleague Dr J. Shlien. Numerous discussions were carried out by the authors with another colleague, Professor G. Dagan, and these left a considerable imprint on this work. The authors thank both.

REFERENCES

Batchelor, G. K. 1967 *An Introduction to Fluid Dynamics*, p. 242. Cambridge University Press.

Proudman, I. A. & Pearson, J. R. A. 1957 *J. Fluid Mech.* **2**, 237.

Schlichting, H. 1960 *Boundary Layer Theory*, 2nd English edn. McGraw-Hill.

Yih, C.-S. 1969 *Fluid Mechanics*, pp. 372–379. McGraw-Hill.

FIGURE 6. Photograph of streamline pattern in the inner field.